Fundamental Interactions

Problem 4: spin, helicity, and Dirac equation The Dirac equation has been introduced as

$$(\gamma^\mu p_\mu - m)\, u(p) = 0 \qquad \text{(particle)} \tag{1a}$$
$$(\gamma^\mu p_\mu + m)\, v(p) = 0 \qquad \text{(anti-particle)} \tag{1b}$$

with

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \,, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \,. \tag{2}$$

Here, $\mathbb{1}_2$ denotes the 2x2 unit matrix and σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \,, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,. \tag{3}$$

The solutions of the Dirac equation are as follows

$$u^{(1,2)}(p) = \sqrt{E+m} \begin{pmatrix} \xi^{(1,2)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\xi^{(1,2)} \end{pmatrix} \,, \qquad v^{(1,2)}(p) = \sqrt{E+m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\xi^{(2,1)} \\ \xi^{(2,1)} \end{pmatrix} \tag{4a}$$

where

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,, \qquad \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \,. \tag{4b}$$

a) Prove that the Pauli matrices are the generators of the Lie algebra $\mathfrak{su}(2)$, i.e. hermitian, traceless matrices that satisfy $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$. Moreover, show that $\sigma_i \cdot \sigma_j = i\varepsilon_{ijk}\sigma_k$ holds for all $i \neq j$.

b) Show that following relation holds:
$$(\vec{\sigma}\cdot\vec{p})\cdot(\vec{\sigma}\cdot\vec{p}) = \vec{p}^2 \,. \tag{5}$$

c) Prove that $u(p)$ and $v(p)$ from (4a) are indeed solutions of the Dirac equation (1).

d) Suppose $\vec{p} = (0, 0, p)$, confirm that the solutions can be re-written as follows:

$$u^{(1)}(p) = \begin{pmatrix} \sqrt{E+m} \\ 0 \\ \sqrt{E-m} \\ 0 \end{pmatrix} \,, \qquad u^{(2)}(p) = \begin{pmatrix} 0 \\ \sqrt{E+m} \\ 0 \\ -\sqrt{E-m} \end{pmatrix} \,, \tag{6a}$$

$$v^{(1)}(p) = \begin{pmatrix} 0 \\ -\sqrt{E-m} \\ 0 \\ \sqrt{E+m} \end{pmatrix} \,, \qquad v^{(2)}(p) = \begin{pmatrix} \sqrt{E-m} \\ 0 \\ \sqrt{E+m} \\ 0 \end{pmatrix} \,. \tag{6b}$$

e) Prove that the four solutions (6) are eigenvectors to the third component of the spin operator

$$\vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \,. \tag{7}$$

Compute the corresponding eigenvalues and provide a physical interpretation for the four solutions (6).

f) Derive expressions for $u^{(1,2)}(p)$ and $v^{(1,2)}(p)$ in the limits: (i) relativistic particle, i.e. $E \gg m$, and (ii) particle at rest, i.e. $E = m$.

The four (fundamental) solutions of the Dirac equation can be interpreted as different spin configurations of the electron and positron. So far, we confirmed this only in the special case of $\vec{p} = (0, 0, p)$. In the following, we will see that the spin is not a "good" quantum number for the Dirac equation.

g) Show that the Hamilton operator of (1) takes the form

$$H = \begin{pmatrix} m \cdot \mathbb{1}_2 & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -m \cdot \mathbb{1}_2 \end{pmatrix} . \tag{8}$$

h) Prove that in the generic case the Hamilton operator H does not commute with the components of the spin operator $\vec{S}$.

i) Show that H commutes with the helicity operator instead, which is defined as

$$\vec{\Sigma} \cdot \vec{p} = \begin{pmatrix} \vec{\sigma} \cdot \vec{p} & 0 \\ 0 & \vec{\sigma} \cdot \vec{p} \end{pmatrix} . \tag{9}$$